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Divisible Effect Algebras

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Divisible effect algebras and their relations to convex effect algebras and MV-algebras are studied. A categorical equivalence between divisible effect algebras and rational vector spaces is proved. Infinitesimal, sharp and extremal elements in divisible effect algebras are studied and their relations to properties of the state space are shown.

KEY WORDS: effect algebras; divisible effect algebras; words, po-groups; rational vector spaces; sharp elements; MV-algebras.

1. INTRODUCTION

Effect algebras as partial algebraic structures with a partially defined operation $\oplus$ and constants 0 and 1 have been introduced as an abstraction of the Hilbert-space effects, i.e., self-adjoint operators between 0 and I on a Hilbert space (Foulis and Bennett, 1994). An alternative structure, D-posets, was introduced by (Kôpka and Chovanec, 1994). These structures play an important role in the theory of quantum measurements, (Bush *et al.*, 1994; Ludwig, 1983).

From the structural point of view, effect algebras are a generalization of boolean algebras, MV-algebras, orthomodular lattices, orthomodular posets, orthoalgebras. For relations among these structures and some other related structures see (e.g., (Pulmannová, 1997; Dvurečenskij and Pulmannová, 2000)).

A very important subclass of effect algebras are so-called interval effect algebras (Bennett and Foulis, 1997), which are representable as intervals of the positive cone in a partially ordered abelian group. Examples of this kind are Hilbert-space effects, MV-algebras, effect algebras with ordering set of states (Bennett and Foulis, 1997), effect algebras with the Riesz decomposition property (Ravindran, 1996; Pulmannová, 1999), convex effect algebras (Gudder and Pulmannová, 1998; Beltrametti *et al.*, 1999). We note that a general characterization of interval effect algebras among effect algebras is an open problem.

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In Pulmannová (2001) it is shown that divisible effect algebras are interval effect algebras. In the proof, "word method" was used, following Baer (1949), Wyler (1966/1967), and Ravindran (1996). This method was compared with the method used for convex effect algebras by Gudder and Pulmannová (1998). Completions in order-unit norm and their relations to state spaces have also been studied there.

In this paper, we prove a categorical equivalence between divisible effect algebras and rational vector spaces. We study relations between sharp and extremal elements in divisible effect algebras. We show that a divisible effect algebra becomes an MV-algebra if and only if it is lattice ordered.

2. EFFECT ALGEBRAS AND INTERVAL EFFECT ALGEBRAS

Let $(E; \oplus, 0, 1)$ be an effect algebra, i.e., $\oplus$ is a partially defined binary operation and 0 and 1 are constants $(0 \neq 1)$, such that the following axioms are satisfied for any a, b, c in E:

(E1) $a \oplus b = b \oplus a$ (commutativity),
(E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity),
(E3) For every $a \in E$ there is a unique $b \in E$ such that $a \oplus b = 1$ (supplementation),
(E4) $a \oplus 1$ is defined iff $a = 0$ (zero-one law).

We denote the element b from (E3) by a', and call it the *orthosupplement* of a. The equalities in (E1) and (E2) are to be understood in such a way that if one side exists, then exists the other and they are equal. Owing to (E2), we need not write brackets in expressions like $a \oplus b \oplus c$ or $a_1 \oplus a_2 \oplus \cdots \oplus a_n$, the latter being defined recurrently.

Define, for $n \in \mathbb{N}$, na as follows: $1a = a$, $na = (n-1)a \oplus a$ if $(n-1)a$ and $(n-1)a \oplus a$ are defined. If there is a greatest $n \in \mathbb{N}$ such that na is defined, then this n is called the *isotropic index* of a, denoted by $\iota(a)$. If na is defined for all $n \in \mathbb{N}$, we put $\iota(a) = \infty$. Define also $0a := 0$ for every $a \in E$.

Let us write $a \perp b$ iff $a \oplus b$ is defined. Then $\perp$ is a binary relation on E, the domain of $\oplus$. Let us also define the binary relation $\leq$ by $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It turns out that the element c is unique, and we shall write $c = b \ominus a$ iff $a \oplus c = b$. Then $\leq$ is a partial order on E. Moreover, $a \perp b$ iff $a \leq b'$. From this it follows that $a \perp b$ and $a_1 \leq a$, $b_1 \leq b$ imply $a_1 \perp b_1$.

In every effect algebra E, the following *cancellation property*: $a \oplus c \leq b \oplus c$ implies $a \leq b$, and *positivity property*: $a \oplus b = 0$ implies $a = 0 = b$ are valid.

Let E, F be effect algebras. A mapping $h : E \to F$ is a *morphism* if $a \perp b$ implies $h(a) \perp h(b)$ and $h(a \oplus b) = h(a) \oplus h(b)$, and $h(1) = 1$. Then clearly, $h(a') = h(a)'$, $h(b) \ominus h(a) = h(b \ominus a)$ whenever $a \leq b$, and $a \leq b$ implies $h(a) \leq h(b)$. A morphism is a *monomorphism* if $a \perp b$ if and only if $h(a) \perp h(b)$. A surjective monomorphism is an *isomorphism*.

Let (G, G^+) be a partially ordered abelian group (additively written) with a positive cone G^+, and choose an element $a \in G^+$. Consider the interval $[0, a] \subseteq G^+$. Define a partial operation $\oplus$ on $[0, a]$ as follows: $x \perp y$ if $x + y \leq a$, and then $x \oplus y = x + y$. It is easy to check that with this operation $\oplus$ and with a as a unit element $[0, a]$ is an effect algebra. A very important class of effect algebras, so called *interval effect algebras*, arise this way (Bennett and Foulis, 1997; Foulis *et al.*, 1994).

Example 2.1. The interval $[0, 1]$ of the real line $\mathbb{R}$ is an interval effect algebra. More generally, let $[0, 1]^X$ be the set of all functions from a set X to the unit interval $[0, 1]$. As an interval of $\mathbb{R}^X$, it is an interval effect algebra. Notice that the above examples are also examples of MV-algebras.

Example 2.2. Let H be a Hilbert space. Consider the group of all bounded self-adjoint operators $\mathcal{B}_s(H)$ on H. The interval $\mathcal{E}(H) := [\theta, I]$, where θ is the zero and I is the identity operator, is an interval effect algebra. Elements of $\mathcal{E}(\mathcal{H})$ are called *Hilbert space effects*.

3. DIVISIBLE EFFECT ALGEBRAS

We say that an effect algebra E is *divisible* if for each $a \in E$ and each $n \in \mathbb{N}$ there is a unique $x \in E$ such that $a = nx$. We will write $x = (1/n)a$.

In the next lemma, we collect some simple properties of divisible effect algebras that have been proved by Pulmannová (2001).

Lemma 3.1. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra.*

(i) *if $m, n \geq 1$ and $a \in E$, then*

$$\frac{1}{n}\left(\frac{1}{m}a\right) = \frac{1}{mn}a.$$

(ii) *If $m, n \geq 2$ and $a \in E$, then $(1/m)a \perp (1/n)a$ and*

$$\frac{1}{m}a \oplus \frac{1}{n}a = (m+n)\left(\frac{1}{mn}a\right) = \frac{m+n}{mn}a.$$

(iii) *If $a, b \in E$ and $a \perp b$ then for any $n \in \mathbb{N}$, $(1/n)a \perp (1/n)b$ and*

$$\frac{1}{n}a \oplus \frac{1}{n}b = \frac{1}{n}(a \oplus b).$$

(iv) *If $a \leq b$, then for any $n \in \mathbb{N}$,*

$$\frac{1}{n}a \leq \frac{1}{n}b.$$

(v) *If* $m \leq n$*, then for any* $a \in E$*,*

$$\frac{1}{n}a \leq \frac{1}{m}a.$$

(vi) *If* na *is defined for* $n \in \mathbb{N}$*,* $a \in E$*, and* $m \geq n$*, then*

$$\frac{1}{m}(na) = n\left(\frac{1}{m}a\right) = \frac{n}{m}a.$$

(vii) *If* na *is defined for some* $n \in \mathbb{N}$*, then for any* $m \in \mathbb{N}$*,* $n((1/m)a)$ *is defined, and*

$$n\left(\frac{1}{m}a\right) = \frac{1}{m}(na).$$

(viii) *If for some* $n \in \mathbb{N}$ *and* $a, b \in E$*,* $(1/n)a = (1/n)b$*, then* $a = b$*.*
(ix) *If for some* $m, n \in \mathbb{N}$ *and* $0 \neq a \in E$*,* $(1/m)a = (1/n)a$*, then* $m = n$*.*
(x) *If* $m, n \geq 2$*, then for any* $a, b \in E$*,* $(1/n)a \perp (1/m)b$*.*

Examples 1 and 2 are divisible effect algebras. An example of an interval effect algebra which is not divisible is so-called diamond (see (Pulmannová, 2001, Example 3)).

Pulmannová (2001) proved the following theorem using the "word" technique following Baer (1949), Wyler (1966/1967) and Ravindran (1996).

Recall that if E is an effect algebra and $(K, +)$ is an abelian group, a morphism $f : E \to K$ such that $a \perp b$, $a, b \in E$ imply $f(a \oplus b) = f(a) + f(b)$ is called a *K-valued measure*.

Theorem 3.2. *Let* $(E; \oplus, 0, 1)$ *be a divisible effect algebra. Then there is a partially ordered abelian group* (G, G^+) *such that* $G = G^+ - G^+$*, with an element* $u \in G^+$ *such that the following properties are satisfied:*

(i) *E is isomorphic with the interval effect algebra* $[0, u]$*,*
(ii) $[0, u]$ *generates* G^+ *(in the sense that every element in* G^+ *is a finite sum of elements of* $[0, u]$*),*
(iii) *every* K*-valued measure* $f : E \to K$ *can be extended uniquely to a group homomorphism* $f^* : G \to K$*.*

The group G from the earlier theorem, which is uniquely defined up to isomorphism, is called a *universal group* (or a *a unigroup*) for E (Foulis and Bennett, 1980). Note that u is an order-unit of G, that is, for every $x \in G$ there is $n \in \mathbb{N}$ such that $-nu \leq x \leq nu$. Moreover, it has been proved that the unigroup can be endowed with a structure of a rational vector space.

Corollary 3.1. *Let $(E; \oplus, 0, 1)$ be a divisible effect algebra. The group G from Theorem 3.2 can be endowed with a structure of an ordered vector space over the field $\mathbb{Q}$ of rational numbers.*

Let (V, V^+) be an ordered rational vector space, and let $u \in V^+$ be an order unit of V. Observe that the the interval $[\theta, u]$ generates V^+. Indeed, for every $x \in V^+$ there is $n \in \mathbb{N}$ such that $\frac{1}{n}x \leq u$, and $x = n(\frac{1}{n}x)$. Moreover, according to Fuchs (1963, Proposition II.3(a)), (V, u) is directed, i.e. $V = V^+ - V^+$.

Recall that a partially ordered abelian group G is *unperforated* if $nx \geq 0$ for some $n \in \mathbb{N}$ and $x \in G$ implies $x \geq 0$ (Goodeaarl, 1986). It is clear that if $(G; u)$ bears a structure of a rational vector space, it must be divisible and unperforated. Conversely, every unperforated partially ordered abelian group can be embedded into an ordered rational vector space via its divisible hull (Pulmannová, 2001).[2]

Recall that an effect algebra E is *convex* (Gudder and Pulmannová, 1998) if for every $a \in E$ and every $\lambda \in [0, 1]$ there exists an element $\lambda a \in E$ such that the following conditions hold.

(C1) If $\alpha, \beta \in [0, 1]$ and $a \in E$, then $\alpha(\beta a) = (\alpha\beta)a$.
(C2) If $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ and $a \in E$, then $\alpha a \perp \beta a$ and $(\alpha + \beta)a = \alpha a \oplus \beta a$.
(C3) If $a, b \in E$ with $a \perp b$ and $\lambda \in [0, 1]$, then $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.
(C4) If $a \in E$, then $1a = a$.

Clearly, every convex effect algebra is divisible.

A map $(\lambda, a) \mapsto \lambda a$ that satisfies (C1)–(C4) is called a *convex structure* on E. Gudder and Pulmannová (1998, Theorem 3.1) showed that every convex effect algebra E is isomorphic (as convex effect algebras) to an interval $[\theta, u]$[3] that generates a real ordered linear space $(V; V^+)$ (which is unique up to order isomorphism), and the effect algebra order coincides with the linear space order restricted to $[\theta, u]$. Clearly, a convex effect algebra is divisible, and it has been proved by Pulmannová (2001) that the ordered rational vector space $(G; G^+)$ from Corollary 1 is a rational subspace of the real linear space $(V; V^+)$ from Theorem 3.1 of Gudder and Pulmannová (1998).

Moreover, it can be proved that every divisible effect algebra can be considered as a subeffect algebra of a convex effect algebra. We need the following lemma.

[2] Consider the direct product $G \times \mathbb{N}$ and an equivalence relation $(a, n) \equiv (b, m)$ if $ma = nb$. Let H be the quotient with respect to this relation. Denote $\frac{a}{n}$ the image of (a, n) in H. If we define operation $+$ on H by $\frac{a}{n} + \frac{b}{m} = \frac{ma+nb}{mn}$, the group H has all required properties. The group H is called the *divisible hull* of G. It is unique up to isomorphism. Putting $H^+ = \{\frac{a}{n} : a \in G^+\}$, we define a partial order on H.

[3] We denote by θ the zero vector of a vector space.

Lemma 3.3. *Let V be a vector space over a field $\mathbb{F}$ and let $\mathbb{G}$ be a field that extends $\mathbb{F}$. Then there is a vector space over a field $\mathbb{G}$ that contains V as an $\mathbb{F}$-subspace.*

Moreover, if $\mathbb{F}$ and $\mathbb{G}$ are ordered with $\mathbb{F}^+ \subset \mathbb{G}^+$, then every ordered vector space (V, V^+) over $\mathbb{F}$ can be embedded into an ordered vector space (W, W^+) over $\mathbb{G}$ as an ordered $\mathbb{F}$-subspace.

Proof: One can consider $\mathbb{G}$ as a vector space over $\mathbb{F}$ and put $W = \mathbb{G} \otimes_{\mathbb{F}} V$. It follows from the universal property of algebraic tensor products that the vector space W over $\mathbb{F}$ can be made into a $\mathbb{G}$-space of the same (Hammel) dimension as V. Indeed, define $\otimes : \mathbb{G} \times V \to \mathbb{G} \otimes_{\mathbb{F}} V$, $(\alpha, v) \mapsto \alpha \otimes v$. The map $f_\alpha : (\beta, v) \mapsto \alpha\beta \otimes v, \alpha, \beta \in \mathbb{G}$, is bilinear, hence induces a unique $\mathbb{F}$-linear map $\tau_\alpha : \mathbb{G} \otimes_{\mathbb{F}} V \to \mathbb{G} \otimes_{\mathbb{F}} V$ such that $\tau_\alpha \cdot \otimes = f_\alpha$. Then the mapping

$$\mathbb{G} \times (\mathbb{G} \otimes_{\mathbb{F}} V) \to \mathbb{G} \otimes_{\mathbb{F}} V,$$
$$(\alpha, x) \mapsto \tau_\alpha(x)$$

induces a $\mathbb{G}$-linear structure on $\mathbb{G} \otimes_{\mathbb{F}} V$. It can be easily verified that $\mathbb{G} \otimes_{\mathbb{F}} V$ becomes a $\mathbb{G}$-linear space.

Now let $\mathbb{F}^+$, $\mathbb{G}^+$, and V^+ be the corresponding positive cones in $\mathbb{F}$, $\mathbb{G}$ and V, and assume that $\mathbb{F}^+ \subset \mathbb{G}^+$. Consider

$$W^+ := \left\{ \sum_{i=1}^{n} \alpha_i \otimes v_i, \alpha_i \in \mathbb{G}^+, v_i \in V^+ \right\}.$$

For every $\beta \in \mathbb{G}^+$, $\tau_\beta\left(\sum_{i=1}^n \alpha_i \otimes v_i\right) = \sum_{i=1}^n \beta\alpha_i \otimes v_i \in W^+$. If $\sum_{i=1}^n \alpha_i \otimes v_i$ and $-\sum_{i=1}^n \alpha_i \otimes v_i$ belong to W^+, then without loss of generality we may assume that $v_i, i \leq n$ are independent. By the definition of W^+, there are β_i in $\mathbb{G}^+$ such that $-\sum_{i=1}^n \alpha_i \otimes v_i = \sum_{i=1}^m \beta_i \otimes v_i$. Now $\sum_{i=1}^n (\alpha_i + \beta_i) \otimes v_i = \theta$, implies $\alpha_i = \beta_i = 0$ since $\alpha_i, \beta_i \geq 0$. Hence, W^+ is an ordering cone in W, and $\mathbb{F}^+ \subset \mathbb{G}^+$ implies $V^+ \subset W^+$. □

Theorem 3.4. *Every divisible effect algebra can be embedded (as a divisible sub-effect algebra) into a convex effect algebra.*

Proof: For a divisible effect algebra E, its unigroup $(G(E), u) =: (V, u)$ is an ordered rational vector space with order unit u. Since $\mathbb{Q} \subset \mathbb{R}$, and $\mathbb{Q}^+ \subset \mathbb{R}^+$, we may use Lemma 2 to construct an ordered real vector space W. It is easy to verify that u is an order unit also for W.

Now E is isomorphic to the interval $[\theta, u]_V$ in V, and the mapping $\gamma : [\theta, u]_V \to [\theta, u]_W$ is an embedding. □

4. CATEGORICAL EQUIVALENCE

In this section, we prove that divisible effect algebras and rational vector spaces with order unit are categorically equivalent. For the basic notions in category theory see e.g. Mac Lane, 1971.

Notice that every morphism $h : E \to F$ of divisible effect algebras preserves also their divisible structure. Indeed, let $0 \leq q = \frac{m}{n} \leq 1$, then $h(\frac{m}{n}a) = mh(\frac{1}{n}a)$, and $h(n(\frac{1}{n}a)) = h(a)$, hence $h(\frac{m}{n}a) = \frac{m}{n}h(a)$. Similarly, we can see that every group homomorphism of rational vector spaces is also a homomorphism in the category of rational vector spaces.

For every divisible effect algebra E, we denote by $(G(E), u)$ its unigroup considered as an ordered rational vector space with order unit u.

Definition 4.3. Let $\mathcal{V}$ be the category whose objects are rational vector spaces with order unit and whose morphisms are homomorphisms of these structures.

Let $\mathcal{D}$ denote the category whose objects are divisible effect algebras and whose morphisms are homomorphisms of these structures.

Define Δ to be the mapping from $\mathcal{D}$ to $\mathcal{V}$ that maps an object E from $\mathcal{D}$ to $(G(E), u)$.

Define Γ to be the mapping from $\mathcal{V}$ to $\Gamma(V, u)$, where $\Gamma(V, u)$ denotes the interval effect algebra $[\theta, u] \subset V^+$.

Lemma 4.5. *For any divisible effect algebra E, ΔE is a rational vector space (V, u) with order unit u, and the embedding map $\alpha : E \to \Delta E$ is a homomorphism of divisible effect algebras. If $f : E \to E_1$ is a homomorphism of divisible effect algebras, then $\Delta f : \Delta E \to \Delta E_1$ is a homomorphism of rational vector spaces with order units. For any rational vector space (V, u) with order unit u, $\Gamma(V, u) = [\theta, u]$ is a divisible effect algebra. If $g : (V, u) \to (V_1, u_1)$ is a homomorphism of rational vector spaces with order units, then $\Gamma g : \Gamma(V, u) \to \Gamma(V_1, u_1)$ is a homomorphism of divisible effect algebras.*

Proof: By Theorem 3.2 and Corollary 3.1, $\Delta E = (G(E), u)$ is a rational vector space with order unit u, and E is isomorphic (as divisible effect algebras) with the unit interval $[\theta, u] \subset G(E)^+$, so α is this isomorphism. If $f : E \to E_1$ is a homomorphism of (divisible) effect algebras, then $f \cdot \alpha : E \to [\theta, u_1]$ is a (V_1, u_1)-valued measure on E, which by Theorem 3.2 extends to a group homomorphism $\Delta f : \Delta E \to \Delta E_1$ which maps u to u_1. Therefore, Δf is also a homomorphism of rational vector spaces with order unit.

If $g : (V, u) \to (V_1, u_1)$ is a homomorphism of rational vector spaces which maps u to u_1, then its restriction $\Gamma g : \Gamma(V, u) \to \Gamma(V_1, u_1)$ that maps $[\theta, u]$ to $[\theta, u_1]$, is clearly a homomorphism of divisible effect algebras. □

From Lemma 4.5 it follows that Δ and Γ are functors. Since ΔE for a divisible effect algebra E is only determined up to isomorphism, the functor Δ is determined up to a natural equivalence $\equiv$.

Theorem 4.6. *The functors $\Delta : \mathcal{D} \to \mathcal{V}$ and $\Gamma : \mathcal{V} \to \mathcal{D}$ form an equivalence of categories that is, $\Gamma \cdot \Delta$ and $\Delta \cdot \Gamma$ are naturally equivalent to the identity functors of $\mathcal{D}$ and $\mathcal{V}$, respectively.*

Proof: Let E be a divisible effect algebra. By Theorem 3.2, $\Gamma(\Delta(E)) = \Gamma(G(E), u) \equiv E$.

Let (V, u) be a rational vector space with order unit u. We may (and will) consider $\Gamma(V, u) = [\theta, u]$ as a divisible effect algebra and construct its unigroup $\Delta\Gamma(V, u) = (G(\Gamma(V, u)), v)$. We have to check that $(G(\Gamma(V, u)), v)$ is isomorphic with (V, u). By Theorem 3.2, the unit interval $[\theta, v]$ in $G(\Gamma(V, u), v)$ is isomorphic with the divisible effect algebra $[\theta, u]$. Denote this isomorphism by $\beta : [\theta, v] \to [\theta, u] \subseteq V$. Then by Theorem 3.2, β uniquely extends to a homomorphism $\beta^* : G(\Gamma(V, u), v) \to (V, u)$. Since (V, u) and $G(\Gamma(V, u)), v)$ are rational vector spaces with order units, the intervals $[\theta, u]$ and $[\theta, v]$ are generating for V^+ and $G(\Gamma(V, u))^+$, respectively and (V, u) and $G(\Gamma(V, u)), v)$ are directed, we have that β^* is an isomorphism.

It follows that $\Gamma \cdot \Delta \equiv I_{\mathcal{D}}$ and $\Delta \cdot \Gamma \equiv I_{\mathcal{V}}$, where $I_{\mathcal{D}}$ and $I_{\mathcal{V}}$ are the identity functors in the categories $\mathcal{D}$ and $\mathcal{V}$, respectively. $\square$

5. INFINITESIMAL, SHARP AND EXTREMAL ELEMENTS

Let V be a rational vector space. A functional $p : V \to \mathbb{R}$ such that for any $x, y \in E$ and any $r \in \mathbb{R}^+$, $p(x + y) \leq p(x) + p(y)$ and $p(rx) = rp(x)$ will be called a *$\mathbb{Q}$-convex functional on E*.

A functional $f : V \to \mathbb{R}$ is *$\mathbb{Q}$-linear* if f is additive and $f(qx) = qf(x)$ for any $q \in \mathbb{Q}$. A $\mathbb{Q}$-linear functional f is *positive* if it maps V^+ into nonnegative numbers.

In what follows, if we speak about a convex (linear) functional on a rational vector space, we mean a $\mathbb{Q}$-convex ($\mathbb{Q}$-linear) functional.

If (V, u) is a rational vector space with order unit u, a *state* on (V, u) is a positive linear functional admitting value 1 on u.

Recall that a divisible effect algebra E is *archimedean* if whenever $a, b, c \in E$ with $a \perp b$ and $c \leq a \oplus n^{-1}b$ for every $n \in \mathbb{N}$, then $c \leq a$ (Pulmannová, 2001). An element $a \in E$ is called *infinitesimal* if a has infinite isotropic index. It follows that an archimedean divisible effect algebra has no nonzero infinitesimals. The converse statement does not hold, in general.

An element $x \in E$ is called an *u-infinitesimal* if $-u \leq nx \leq u$ for every $n \in \mathbb{N}$.

Define the mapping on (V, u) as follows:

$$\|x\|_u = \inf\{q \in \mathbb{Q}^+ : -qu \leq x \leq qu\}.$$

According to Goodearl (1986), this mapping has the following norm-like properties (we write $\|\cdot\|$ instead of $\|\cdot\|_u$ if no confusion threatens):

(a) $\|mx\| = m\|x\|$ $(m \in \mathbb{N})$,
(b) $\|x + y\| \leq \|x\| + \|y\|$,
(c) if $-y \leq x \leq y$, then $\|x\| \leq \|y\|$.

If V is nonzero, then

(d) $\|u\| = 1$,
(e) $\|x\| = \max\{|s(x)| : s \in S(V, u)\}$, where $\mathcal{S}(V, u)$ denotes the state space of (V, u).

A state on an effect algebra E is a morphism ω from E into the effect algebra $[0, 1] \subseteq \mathbb{R}$. We denote the set of all states on E by $\mathcal{S}(E)$. It can be shown that every interval effect algebra has at least one state (Bennett and Foulis, 1997), and hence every divisible effect algebra possesses at least one state. However, even if E is divisible, it may have only one state (see e.g. Beltrametti *et al.*, 1999, Example 2), and in applications it is important to have a rich supply of states. We say that $S \subseteq \mathcal{S}(E)$ is *separating* if $\omega(a) = \omega(b)$ for every $\omega \in S$ implies that $a = b$. We say that $S \subseteq \mathcal{S}(E)$ is *ordering* (or *order determining*) if $\omega(a) \leq \omega(b)$ for every $\omega \in S$ implies $a \leq b$. If S is ordering, then S is separating. The converse need not hold. The role of the properties of the state space can be seen from the following theorems, which are analogous to corresponding theorems in convex effect algebras (Beltrametti *et al.*, 1999).

Theorem 5.7. *Let E be a divisible effect algebra, $(G(E), u)$ its unigroup. The following statements are equivalent:*

(i) *E possesses a separating set of states,*
(ii) *$\|\cdot\|_u$ is actually a norm,*
(iii) *$(G(E), u)$ has no u-infinitesimals.*

Theorem 5.8. *Let E be a divisible effect algebra, $(G(E), u)$ its unigroup. The following statements are equivalent:*

(i) *E possesses an ordering set of states.*
(ii) *E is archimedean.*
(iii) *$(G(E), u)$ is archimedean.*
(iv) *E can be embedded into a unit interval of an archimedean real order unit space.*

Recall that an element in an effect algebra is *sharp* if $a \wedge a'$ exists and equals 0.

Let E be a divisible effect algebra. We say that an element a in E is *extremal* if for any $\lambda \in (0, 1) \cap \mathbb{Q}$, $a = \lambda b + (1 - \lambda)c$ implies $b = c$. Similarly, as in the case of convex effect algebras (Beltrametti *et al.*, 1999), we obtain that every extremal element is sharp, and that if $a \neq 0$ is sharp and $a = \lambda b \oplus (1 - \lambda)c, \lambda \in (0, 1) \cap \mathbb{Q}$, then b and c are sharp.

Let E be an effect algebra. In agreement with (Beltrametti *et al.*, 1999), we will say that the set S, $S \subseteq \mathcal{S}(E)$ is *sharply determining* if for any sharp $a \in E$ and any $b \in E$ with $a \not\leq b$ there is $\omega \in S$ such that $\omega(a) = 1$ and $\omega(b) \neq 1$.

Theorem 5.9. *Let E be a divisible effect algebra with a sharply determining set of states $S \subseteq \mathcal{S}(E)$. Then the following cases result:*

(a) *Every sharp element of E is extremal, and hence the sharp and extremal elements coincide.*
(b) *The set of all sharp elements forms a subeffect algebra E_s of E. Moreover, with respect to the operations taken in it, E_s is an orthomodular poset.*

Proof: The proofs of (a) and the first part of (b) are the same as for convex effect algebras, see (Beltrametti *et al.*, 1999), we will prove only the last statement in (b).

Assume that $a, b \in E_s$, $a \perp b$, then $c := a \oplus b \in E_s$ and $a \leq c$, $b \leq c$. Assume that $d \in E_s$ is an upper bound of a, b. Then d' is a lower bound of a', b', hence $\omega(d') = 1$ implies $\omega(a') = \omega(b') = 1$. This yields $\omega(a) = \omega(b) = 0$, hence $\omega(a \oplus b) = 0$. It follows that $\omega(d') = 1$ implies $\omega((a \oplus b)') = 1$, hence $a \oplus b \leq d$. We proved that $a \oplus b$ is the least upper bound of a and b in E_s, i.e., $a \oplus b = a \vee_{E_s} b$, which means that E_s bears a structure of an OMP. □

6. RELATIONS TO MV-ALGEBRAS

An MV-algebra is an algebraic system $(A, \uplus, ', 0, 1)$, where $0, 1 \in A$, $'$ is a unary operation and $\uplus$ is a binary operation on A, which satisfy the following conditions:

(MV1) $(a \uplus b) \uplus c = a \uplus (b \uplus c)$,
(MV2) $a \uplus 0 = a$,
(MV3) $a \uplus b = b \uplus a$,
(MV4) $a \uplus 1 = 1$,
(MV5) $a \uplus a' = 1$
(MV6) $(a')' = a$,
(MV7) $0' = 1$,
(MV8) $(a' \uplus b)' \uplus b = (a \uplus b')' \uplus a$.

The concept of MV-algebras was introduced to study multi-valued logics (Chang, 1958; Mundici, 1986). As with effect algebras, a partial order can be introduced on A by defining $a \leq b$ if there exists a $c \in A$ such that $a \uplus c = b$. It can be shown that A is a distributive lattice with respect to this ordering. We write $a \perp b$ if $a \leq b'$. Relationships between effect algebras (or, equivalently, D-posets) and MV-algebras have been studied by Kôpka and Chovanec (1994), Pulmannová (1997), and Dvurecčenskij and Pulmannová (2000).

We say that an effect algebra $(E; \oplus, 0, 1)$ is an MV-algebra if there is an operation $\uplus$ such that $(E; \uplus, ', 0, 1)$ is an MV-algebra and $a \oplus b = a \uplus b$ whenever $a \perp b$. Hence, if E is an MV-algebra, E is lattice-ordered.

Conversely, given an MV-algebra A, we can endow A with a structure of an effect algebra with the same order by restricting the total $\uplus$-operation to orthogonal pairs.

According to Mundici (1986), a unit interval $[0, u]$ in an abelian lattice-ordered group (G, u) with order unit u is an MV-algebra, and conversely, every MV-algebra A can be represented this way. Moreover, (G, u) is a unigroup for A if we consider it as an effect algebra (Ravindran, 1996).

Observe that in a lattice ordered divisible effect algebra we have $\frac{1}{n}a \vee \frac{1}{n}b = \frac{1}{n}(a \vee b)$ $(a, b \in E, n \in \mathbb{N})$. Indeed,

$$\frac{1}{n}a, \frac{1}{n}b \leq \frac{1}{n}(a \vee b) \leq \frac{1}{n}1.$$

Assume that for a $c \in E$, we have $\frac{1}{n}a, \frac{1}{n}b \leq c$. Then

$$\frac{1}{n}a, \frac{1}{n}b \leq c \wedge \left(\frac{1}{n}1\right) \leq \frac{1}{n}1 \Rightarrow a, b \leq n\left(c \wedge \left(\frac{1}{n}1\right)\right) \leq 1.$$

Hence,

$$a \vee b \leq n\left(c \wedge \left(\frac{1}{n}1\right)\right), \Rightarrow \frac{1}{n}(a \vee b) \leq c \wedge \left(\frac{1}{n}1\right) \leq c.$$

Similarly, as for convex effect algebras, we have the following statement. The proof follows the pattern of convex effect algebras.

Theorem 6.10. *A divisible effect algebra E is an MV-algebra if and only if E is lattice ordered.*

It is well known that the divisible hull of a lattice-ordered abelian group is lattice-ordered, which yields the following result.

Theorem 6.11. *Every MV-algebra can be embedded into a unit interval $[\theta, u]$ of a rational vector lattice (V, u) with order unit u.*

Recall that an effect algebra E has the *Riesz decomposition property* (RDP, in short) if the following equivalent conditions are satisfied.

(1) $a \leq b \oplus c\ (a, b, c \in E) \Rightarrow\ a = a_1 \oplus a_2, a_1, a_2 \in E, a_1 \leq b, a_2 \leq c.$
(2) $a_1 \oplus a_2 = b_1 \oplus b_2\ (a_i, b_j \in E, i, j \in \{1, 2\}\ \Rightarrow\ \exists w_{ij} \in E, i, j \in \{1, 2\},$
$a_i = w_{i1} \oplus w_{i2}, i \in \{1, 2\}, b_j = w_{1j} \oplus w_{2j}, j \in \{1, 2\}.$

Notice that a lattice-ordered effect algebra satisfies RDP if and only if it is an MV-algebra. There are lattice-ordered effect algebras which do not satisfy RDP, e.g. the "diamond" (Dvurečenskij and Pulmannová, 2000).

Recall that an abelian partially ordered group (G, G^+) has the Riesz decomposition property if one of the (equivalent) conditions (1) and (2) is satisfied for positive elements with $\oplus$ replaced by the group operation $+$.

Notice that every lattice-ordered abelian group satisfies RDP. It has been proved by Ravindran (1996) that every effect algebra E with RDP is isomorphic with a unit interval $[0, u]$ in its unigroup (G, u), which also has RDP. Moreover, following Fuchs (1965, Proposition 2.5), an unperforated abelian group with the RDP can be embedded into a divisible hull which has also the RDP. Summarizing, we obtain the following result.

Theorem 6.12. *(a) Every divisible effect algebra E with the Riesz decomposition property is isomorphic with an interval* $[\theta, u]$ *in a rational vector space* (V, u) *with RDP.*

(b) An effect algebra E with the Riesz decomposition property can be embedded into a divisible effect algebra with RDP if and only if its unigroup $(G(E), u)$ *is unperforated.*

We will say that an element a in a divisible MV-algebra A is extremal if it is extremal in the effect algebra corresponding to A.

Our last result shows that sharp and extremal elements in a divisible MV-algebra coincide.

Theorem 6.13. *An element a in a divisible MV-algebra A is sharp if and only if it is extremal.*

Proof: It suffices to prove that a sharp element is extremal. Assume that $a \in A$ is sharp, and $a = \lambda b \oplus (1 - \lambda)c$, $\lambda \in [0, 1] \cap \mathbb{Q}$. Then b and c are sharp. It is easy to check that $a' = \lambda b' \oplus (1 - \lambda)c'$. Assume that there is a nonzero element $x \in A$ such that $x \leq b, x \leq c'$. We find $n \in \mathbb{N}$ such that $\frac{1}{n}$ is a lower bound of λ and $(1 - \lambda)$. Then $\frac{1}{n}x \leq \lambda b \leq a$, and $\frac{1}{n}x \leq (1 - \lambda)c' \leq a'$. It follows that $x = 0$, hence $b \wedge c' = 0$. But in an MV-algebra $b \wedge c' = 0$ implies $b \perp c'$ (Pulmannová, 1997), hence $b \leq c$. By symmetry, so $b = c$. This proves that a is extremal. □

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